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The Distribution of Zeros and Poles of Asymptotically Extremal Rational Functions for Zolotarev's Problem

A. L. Levin

Department of Mathematics, The Open University of Israel, P.O. box 39328, Tel Aviv, Israel
E-mail: elile@oumail.openu.ac.il

and

E. B. Saff[1]

Institute for Constructive Mathematics, Department of Mathematics, University of South Florida, Tampa, Florida 33620, U.S.A.
E-mail: esaff@math.usf.edu

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We investigate the possible limit distributions of zeros and poles associated with ray sequences of rational functions that are asymptotically optimal for weighted Zolotarev problems. For disjoint compacta E_1, E_2 in the complex plane, the Zolotarev problem entails minimizing the ratio of the sup over E_1 of the modulus of a weighted rational to its inf over E_2. Potential theoretic tools are utilized in the analysis.

Key Words: Zolotarev problem; rational functions; asymptotic distributions; zeros and poles; potential theory.

1. INTRODUCTION

Let E_1, E_2 be closed sets in the complex plane $\mathbf{C}$ that are a positive distance apart. Given a pair (m, n) of non-negative integers, denote by $\mathbf{R_{mn}}$ the class of all rational functions in the complex variable z whose numerator and denominator degrees are m and n, respectively. Let w be an

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admissible weight (see Definition 2.1) on $E_1 \cup E_2$. Given $r \in \mathbf{R_{mn}}$, consider the quantity

$$Z_{mn}(r; w) := \frac{\sup_{z \in E_1}\{|r(z)|\, w(z)^{m+n}\}}{\inf_{z \in E_2}\{|r(z)|\, w(z)^{m+n}\}} \tag{1.1}$$

and set

$$Z_{mn}(w) := \inf\{Z_{mn}(r; w) \colon r \in \mathbf{R_{mn}}\}. \tag{1.2}$$

We call $Z_{mn}(w)$ the *weighted Zolotarev constant* of type (m, n) for the condenser (E_1, E_2). Next, fix $0 < \tau < 1$ and consider a "ray sequence" of pairs (m, n), namely a sequence N_τ for which

$$m+n \to \infty, \qquad \frac{m}{n} \to \frac{\tau}{1-\tau}. \tag{1.3}$$

Our first task will be to show that for any such N_τ,

$$\lim_{(m,n) \in N_\tau} \{Z_{mn}(w)\}^{1/(m+n)} = \exp(-F_{w,\tau}), \tag{1.4}$$

where $F_{w,\tau}$ is a quantity that arises in the solution to a certain energy problem discussed in Section 2. This generalizes a previous result obtained by the authors [LeSa].

Next, to each

$$r_{mn}(z) = \frac{\prod_{i=1}^{m} (z - \alpha_{im})}{\prod_{i=1}^{n} (z - \beta_{in})} \tag{1.5}$$

we associate the normalized distribution

$$\nu(r_{mn}) := \frac{1}{m+n} \left\{ \sum_{i=1}^{m} \delta_{\alpha_{im}} - \sum_{i=1}^{n} \delta_{\beta_{in}} \right\}, \tag{1.6}$$

where, generically, δ_z stands for the point distribution with total mass 1 at z. Given a ray sequence N_τ (cf. (1.3)), we call the sequence $\{r_{mn}\}$, $r_{mn} \in \mathbf{R_{mn}}$, *asymptotically extremal* if

$$\lim_{(m,n) \in N_r} \{Z_{mn}(r_{mn}; w)\}^{1/(m+n)} = \exp(-F_{w,\tau}). \tag{1.7}$$

We shall see, in the course of the proof of (1.4), that there exists such $\{r_{mn}\}$ for which $\nu(r_{mn})$ converges to the equilibrium distribution μ^* for the above mentioned energy problem. The convergence is understood in the weak-star sense on the Riemann sphere $\overline{\mathbf{C}}$, that is we write $\mu_n \to \mu$ if $\int f\, d\mu_n \to \int f\, d\mu$, for every f that is continuous on $\overline{\mathbf{C}}$. Simple examples show that there may be other asymptotically extremal sequences $\{r_{mn}\}$ for which $\nu(r_{mn}) \to \mu \neq \mu^*$.

The main objective of this paper is to describe all possible weak-star limits of sequences $\{\nu(r_{mn})\}$ associated with asymptotically extremal sequences $\{r_{mn}\}$. We will concentrate on the case of *bounded* E_1, E_2. The case of unbounded sets requires additional assumptions on the weight. These are briefly discussed at the end of the paper.

2. TWO EXTREMAL PROBLEMS OF POTENTIAL THEORY

We take the weight in the form $w = \exp(-Q)$, where Q is a function from $E_1 \cup E_2$ to the extended real line $[-\infty, \infty]$.

DEFINITION 2.1. Let E_1, E_2 be disjoint compacta in $\mathbf{C}$, both of positive logarithmic capacity. A weight $w = \exp(-Q)$ is called *admissible* if the following conditions hold:

(i) Q is a lower (upper) semicontinuous function on E_1 (on E_2)

(ii) $Q < \infty$ $(Q > -\infty)$ on a subset of E_1 (of E_2) that has positive logarithmic capacity.

We remark that a lower (upper) semicontinuous function does not attain, by definition, the value $-\infty$ $(+\infty)$.

Given $0 < \tau < 1$, let M_τ denote the set of all signed measures $\mu = \mu_1 - \mu_2$ that have a compact support in $\mathbf{C}$ and satisfy $\|\mu_1\| = \tau$, $\|\mu_2\| = 1 - \tau$. If, additionally, $S(\mu_i) \subset E_i$, $i = 1, 2$, we write $\mu \in M_\tau(E_1, E_2)$. Here and throughout, S stands for the support of indicated measure.

With the usual notation

$$U^\mu(z) := \int \log \frac{1}{|z-t|}\, d\mu(t),$$

$$I(\mu) := \int U^\mu \, d\mu = \iint \log \frac{1}{|z-t|}\, d\mu(z)\, d\mu(t),$$

consider the following extremal problems:

$$V = V_{w,\tau} := \inf \left\{ I(\mu) + 2 \int Q\, d\mu : \mu \in M_\tau(E_1, E_2) \right\} \tag{2.1}$$

$$F = F_{w,\tau} := \sup \left\{ \text{“}\inf_{E_1}\text{”}(U^\mu + Q) - \text{“}\sup_{E_2}\text{”}(U^\mu + Q) : \mu \in M_\tau \right\}, \tag{2.2}$$

where "inf" and "sup" mean, respectively, inf and sup neglecting sets of zero capacity.

The following result is well-known (cf. [SaTo, p. 383]):

THEOREM 2.2. *Let E_1, E_2, and w be as above. Then for any $0<\tau<1$,*

(i) *$V_{w,\tau}$ is finite and there exists a unique*

$$\mu^* = \mu^*_{w,\tau} = \mu_1^* - \mu_2^* \in M_\tau(E_1, E_2)$$

for which $V_{w,\tau} = I(\mu^*) + 2\int Q\,d\mu^*$.

(ii) *μ^* has finite logarithmic energy and both U^{μ^*} and Q are bounded on $S(\mu^*)$. Consequently, U^{μ^*} is bounded on compact subsets of $\mathbf{C}$.*

(iii) *There exist constants F_1, F_2 (depending on w, τ) such that*

$$U^{\mu^*} + Q \leqslant F_1 \quad \text{on } S(\mu_1^*), \qquad U^{\mu^*} + Q \geqslant F_1 \quad \text{q.e. on } E_1,$$

$$U^{\mu^*} + Q \geqslant -F_2 \quad \text{on } S(\mu_2^*), \qquad U^{\mu^*} + Q \leqslant -F_2 \quad \text{q.e. on } E_2$$

(here and throughout q.e. means neglecting sets of zero capacity). Consequently,

$$U^{\mu^*} + Q = \begin{cases} F_1 & \text{q.e. on } S(\mu_1^*) \\ -F_2 & \text{q.e. on } S(\mu_2^*). \end{cases} \tag{2.3}$$

(iv) *For any $\mu \in M_\tau$,*

$$\underset{E_1}{\text{“inf”}}(U^\mu + Q) - \underset{E_2}{\text{“sup”}}(U^\mu + Q) \leqslant F_1 + F_2. \tag{2.4}$$

We see from Theorem 2.2(iii), (iv) that the value $F_{w,\tau}$ in problem (2.2) is equal to $F_1 + F_2$ and it is attained for $\mu = \mu^*$. However, an extremal measure for this problem may be not unique (see Examples 2.4, 2.5 below); hence we introduce

DEFINITION 2.3. Given $\mu \in M_\tau$, we say that $\mu \in M_\tau^* = M_\tau^*(w)$ if

$$\underset{E_1}{\text{“inf”}}(U^\mu + Q) - \underset{E_2}{\text{“sup”}}(U^\mu + Q) = F_1 + F_2.$$

We adopt the simplified notation

$$F_1(\mu) := \underset{E_1}{\text{“inf”}}(U^\mu + Q), \qquad -F_2(\mu) := \underset{E_2}{\text{“sup”}}(U^\mu + Q), \tag{2.5}$$

so that the above definition takes the form

$$M_\tau^* = M_\tau^*(w) = \{\mu \in M_\tau : F_1(\mu) + F_2(\mu) = F_1 + F_2\} \tag{2.6}$$

(where $F_i = F_i(\mu^*)$, $i = 1, 2$).

EXAMPLE 2.4. Let E_1 be the circle $|z| = 1$ and let E_2 be the union of the circles $|z| = R_1$, $|z| = R_2$, $1 < R_1 < R_2$. Assuming $\tau < 1/2$, take any α that satisfies

$$0 \leqslant \alpha \leqslant \frac{\tau}{1-\tau} \qquad (<1) \tag{2.7}$$

and consider $\mu = \mu_1 - \mu_2 \in M_\tau$ that is defined by

$$\begin{aligned} \mu_1 &= \tau \frac{1}{2\pi} \, d\theta \qquad \text{on} \quad |z| = 1, \\ \mu_2 &= (1-\tau) \frac{1}{2\pi} \begin{cases} \alpha \, d\theta & \text{on} \quad |z| = R_1 \\ (1-\alpha) \, d\theta & \text{on} \quad |z| = R_2 \end{cases} \end{aligned} \tag{2.8}$$

($d\theta$ stands for the angular measure on the indicated circle). Simple calculation gives (we take $Q \equiv 0$ in this example)

$$\inf_{E_1} U^\mu - \sup_{E_2} U^\mu = \tau \log R_1. \tag{2.9}$$

On the other hand, the extremal measure $\mu^* = \mu_1^* - \mu_2^*$, being unique, must have the form (2.8), for some $\alpha \in [0, 1]$. Utilizing Theorem 2.2(iii), we find that this α is equal to $\tau/(1-\tau)$. Therefore (2.7) to (2.9) show that any μ that is given by (2.7), (2.8) belongs to M_τ^*. Note that $S(\mu) = S(\mu^*)$ if $\alpha \neq 0$, while $S(\mu_1) = S(\mu_1^*)$, $S(\mu_2) \subset S(\mu_2^*)$ if $\alpha = 0$.

Our next example shows that there may be $\mu \in M_\tau^*$ for which $S(\mu_2)$ intersects E_1 or even intersects Int E_1 (the interior of E_1).

EXAMPLE 2.5. Let E_1 be the union of the circle $|z| = r < 1$, and the set $R \leqslant |z| \leqslant 2R$, $R > 1$. Let E_2 be the circle $|z| = 1$. Again, we take $Q \equiv 0$. It can be shown (cf. [LeSa, Example 5.2], where $|z| = R$ rather than $R \leqslant |z| \leqslant 2R$ was considered) that for

$$\tau < \tau_0 := \frac{\log R}{\log R + \log(R/r)} \qquad \left(<\frac{1}{2}\right)$$

we have $F_1 + F_2 = \tau \log(1/r)$, while μ^* is given by

$$\mu_1^* = \tau \frac{1}{2\pi}\, d\theta \qquad \text{on} \quad |z| = r,$$

$$\mu_2^* = (1-\tau)\frac{1}{2\pi}\, d\theta \qquad \text{on} \quad |z| = 1.$$

Define $\mu \in M_\tau$ by

$$\begin{aligned} \mu_1 &= \tau \frac{1}{2\pi}\, d\theta \qquad \text{on} \quad |z| = r, \\ \mu_2 &= (1-\tau)\frac{1}{2\pi}\begin{cases} \alpha\, d\theta & \text{on} \quad |z| = 1 \\ (1-\alpha)\, d\theta & \text{on} \quad |z| = R_1, \quad R_1 \geqslant R \end{cases} \end{aligned} \tag{2.10}$$

with α satisfying

$$1 > \alpha \geqslant \frac{\tau}{1-\tau}\,\frac{1-\tau_0}{\tau_0}. \tag{2.11}$$

Calculation shows that the corresponding potential satisfies

$$U^\mu(z) = \begin{cases} \tau \log \dfrac{1}{r} - (1-\tau)(1-\alpha)\log \dfrac{1}{R_1}, & |z| = r \\ \tau \log \dfrac{1}{|z|} - (1-\tau)\,\alpha \log \dfrac{1}{|z|} - (1-\tau)(1-\alpha)\log \dfrac{1}{R_1}, & |z| \geqslant R \\ -(1-\tau)(1-\alpha)\log \dfrac{1}{R_1}, & |z| = 1. \end{cases}$$

The condition (2.11) ensures that the value of U^μ on $|z| = r$ does not exceed any of its values on $|z| \geqslant R$. Therefore,

$$F_1(\mu) + F_2(\mu) = \tau \log \frac{1}{r},$$

so that $\mu \in M_\tau^*$.

Note that $S(\mu_2)$ has points on the boundary of E_1 (if $R_1 = R$, say) or in the interior of E_1 (if $R < R_1 < 2R$). In the latter case no asymptotically extremal sequence $\{r_{mn}\}$ exists, for which $v(r_{mn}) \to \mu$ (otherwise, r_{mn} would have poles on E_1, which would imply $Z_{mn} = \infty$). Note also, that as $R_1 \to \infty$, the corresponding part of μ_2 converges to a point mass at ∞. Since the measures in M_τ^* are supposed to have a compact support, we see that M_τ^* may not be closed in the weak-star topology.

Finally, we could introduce here a weight by setting $Q=0$ for $|z|=r$ and for $|z|=1$, and by choosing Q to be an arbitrary *positive* lower semicontinuous function for $R \leqslant |z| \leqslant 2R$. Since the value of the original U^{μ^*} on $|z|=r$ did not exceed any of its values on $R \leqslant |z| \leqslant 2R$, we easily deduce that, for the new problem, μ^*, F_1, F_2 remain the same. Therefore μ that is given by (2.10), (2.11) is again in M_τ^*, but now Q may be unbounded on $S(\mu_2) \cap \operatorname{Int} E_1$. Compare this with the assertion (ii) of Theorem 2.2, according to which Q must be bounded on $S(\mu^*)$.

3. ASYMPTOTICS OF $Z_{mn}(w)$

THEOREM 3.1. *Let w be an admissible weight on $E_1 \cup E_2$, and let N_τ be a ray sequence* (*cf.* (1.3)). *Then*

$$\lim_{(m,n) \in N_\tau} \{Z_{mn}(w)\}^{1/(m+n)} = \exp(-F_{w,\tau}). \tag{3.1}$$

Moreover, let $\mu = \mu_1 - \mu_2 \in M_\tau^*$ be such that

$$S(\mu_1) \cap E_2 = \varnothing, \qquad S(\mu_2) \cap E_1 = \varnothing. \tag{3.2}$$

(*In particular, this holds for $\mu = \mu^*$.*) *Then an asymptotically extremal sequence $\{r_{mn}\}$, $(m,n) \in N_\tau$, exists for which*

$$\nu_{mn} := \nu(r_{mn}) \to \mu, \qquad (m,n) \in N_\tau. \tag{3.3}$$

Proof. This follows the same lines as the proof in [LeSa, Sect. 6], so we will be brief.

For any $r \in \mathbf{R_{mn}}$ we have (recall (1.1), (1.5), (1.7), (2.4))

$$\begin{aligned} -\frac{1}{m+n} \log Z_{mn}(r; w) &= \inf_{E_1}(U^{\nu_{mn}} + Q) - \sup_{E_2}(U^{\nu_{mn}} + Q) \\ &\leqslant \text{“}\inf_{E_1}\text{”}(U^{\nu_{mn}} + Q) - \text{“}\sup_{E_2}\text{”}(U^{\nu_{mn}} + Q) \\ &\leqslant F_{w,\tau_{mn}}, \end{aligned}$$

where

$$\tau_{mn} := \frac{m}{m+n} \to \tau, \qquad (m,n) \in N_\tau,$$

by (1.3).

Since $F_{w,\tau}$ is a concave (therefore continuous) function of τ on $(0, 1)$ (the proof is the same as in [LeSa, p. 242]), we conclude that for any sequence N_τ there holds

$$\liminf_{(m,n)\in N_\tau} \{Z_{mn}(w)\}^{1/(m+n)} \geqslant \exp(-F_{w,\tau}). \tag{3.4}$$

Next, since $\mu \in M_\tau^*$, we have $F_1(\mu)+F_2(\mu)=F_{w,\tau}$, and the exceptional sets

$$E_1(\varepsilon) := \{z \in E_1\colon U^\mu(z)+Q(z) \leqslant F_1(\mu)-\varepsilon\}$$

$$E_2(\varepsilon) := \{z \in E_2\colon U^\mu(z)+Q(z) \geqslant -F_2(\mu)+\varepsilon\}$$

have zero capacity, for any $\varepsilon > 0$.

Let $\sigma_\varepsilon \in M_\tau(E_1, E_2)$ be such that $U^{\sigma_\varepsilon}=+\infty$ on $E_1(\varepsilon)$, $U^{\sigma_\varepsilon}=-\infty$ on $E_2(\varepsilon)$. Then, given any $0<\alpha<1$, one can construct a sequence $\{v_{mn}\} \in M_{\tau_{mn}}$,

$$v_{mn}=\frac{1}{m+n}\left\{\sum_{i=1}^{m} \delta_{\alpha_{im}} - \sum_{i=1}^{n} \delta_{\beta_{in}}\right\} =: v_{mn,1}-v_{mn,2},$$

where $\alpha_{im} \in E_1(\varepsilon) \cup S(\mu_1)$, $\beta_{in} \in E_2(\varepsilon) \cup S(\mu_2)$, and such that

$$v_{mn} \to (1-\alpha)\,\mu+\alpha\sigma_\varepsilon, \qquad (m,n)\in N_\tau.$$

By (3.2), we obtain that $U^{v_{mn,1}} \to U^{(1-\alpha)\mu_1+\alpha\sigma_{\varepsilon,1}}$ on E_2, and $U^{v_{mn,2}} \to U^{(1-\alpha)\mu_2+\alpha\sigma_{\varepsilon,2}}$ on E_1. Using these relations, the principle of descent and semi-continuity property of Q, one obtains as in [LeSa] that

$$\limsup_{(m,n)\in N_\tau} (Z_{mn}(r_{mn}, w))^{1/(m+n)} \leqslant \exp(-F_{w,\tau}(\alpha,\varepsilon)),$$

where

$$F_{w,\tau}(\alpha,\varepsilon) := (1-\alpha)\,F_{w,\tau}-2(1-\alpha)\,\varepsilon-\alpha c(\varepsilon)$$

and $r_{mn} \in \mathbf{R_{mn}}$ is constructed by (1.5).

Now, on first letting $\alpha \to 0$ and then $\varepsilon \to 0$ and utilizing the standard diagonal procedure, one can construct some sequence N_τ of the form (1.3) such that $v_{mn} \to \mu$, $(m,n)\in N_\tau$, while

$$\limsup_{(m,n)\in N_\tau} \{Z_{mn}(r_{mn}; w)\}^{1/(m+n)} \leqslant \exp(-F_{w,\tau}).$$

Together with (3.4) this proves (3.1) as well as the fact that r_{mn} is a desired sequence. The passage from *some* N_τ to *any* N_τ is simple (cf. [LeSa, pp. 255–256]). ∎

In many cases one can relax (3.2) to the condition

$$S(\mu_1) \cap \operatorname{Int} E_2 = \varnothing, \qquad S(\mu_2) \cap \operatorname{Int} E_1 = \varnothing. \tag{3.5}$$

The idea is as follows. Given a set D and $\varepsilon > 0$, let

$$D^\varepsilon := \{z: \operatorname{dist}(z, D) < \varepsilon\} \tag{3.6}$$

denote the open ε-neighborhood of D.

Take $\varepsilon > 0$ small enough so that $E_1^\varepsilon \cap E_2^\varepsilon = \varnothing$. Then, replace that part of μ_1 (of μ_2) that sits on E_2^ε (on E_1^ε) by its balayage on the boundary ∂E_2^ε (∂E_1^ε). Let μ_1^ε, μ_2^ε be the resulting measures. By the known properties of balayage, we have that $\mu^\varepsilon := \mu_1^\varepsilon - \mu_2^\varepsilon \in M_\tau$, while

$$F_1(\mu^\varepsilon) + F_2(\mu^\varepsilon) \geqslant F_1(\mu) + F_2(\mu).$$

Since $\mu \in M_\tau^*$, we obtain (see Theorem 2.2(iv)) that $\mu^\varepsilon \in M_\tau^*$ as well, but now μ^ε satisfies (3.2). If we knew that $\mu^\varepsilon \to \mu$ as $\varepsilon \to 0$, we could apply Theorem 3.1 to μ^ε and then, on letting $\varepsilon \to 0$ and using he diagonal procedure, we could construct the desired asymptotically extremal sequence $\{r_{mn}\}$.

Obviously, (3.5) is necessary for $\mu^\epsilon \to \mu$. Following are several conditions each of which ensures that $\mu^\epsilon \to \mu$.

THEOREM 3.2. *Let μ^ε be as above. Assume that* (3.5) *holds and, in addition, one of the following conditions is satisfied*:

(i) *We have*

$$\mu_1(\partial(\operatorname{Int} E_2)^I) = \mu_2(\partial(\operatorname{Int} E_1)^I) = 0, \tag{3.7}$$

where $\partial(\operatorname{Int} E_i)^I \subset \partial E_i$ denotes the set of irregular points of all components of $\operatorname{Int} E_i$, $i = 1, 2$.

(ii) *U^{μ_1} is bounded on every component of* $\operatorname{Int} E_2$, *while U^{μ_2} is bounded on every component of* $\operatorname{Int} E_1$.

(iii) *Q is bounded on the set $\partial E_1 \cup \partial E_2$.*

Then $\mu^\varepsilon \to \mu$ as $\varepsilon \to 0$ and, consequently, the second assertion of Theorem 3.1 *holds true.*

COROLLARY 3.3. *Assume*

$$\operatorname{Int} E_1 = \operatorname{Int} E_2 = \varnothing. \tag{3.8}$$

Then the second assertion of Theorem 3.1 *holds for any $\mu \in M_\tau^*$.*

COROLLARY 3.4. *If every component of* Int E_i, $i=1, 2$, *is regular, then the second assertion of Theorem* 3.1 *holds for any* $\mu \in M_\tau^*$ *that satisfies* (3.5).

COROLLARY 3.5. *If* Q *is bounded on* $E_1 \cup E_2$, *then the condition* (3.5) *is necessary and sufficient for the second assertion of Theorem* 3.1 *to be true.*

These corollaries are obvious. Concerning the necessity of (3.5) in Corollary 3.5, we refer to the end of Section 2.

Proof of Theorem 3.2. We will show that our assumptions imply $\mu_1^\varepsilon \to \mu_1$ as $\varepsilon \to 0$. The proof of $\mu_2^\varepsilon \to \mu_2$ is similar. Let $\varepsilon_n \downarrow 0$. Then $\{U^{\mu_1^{\varepsilon_n}}\}$ is an increasing sequence majorized by U^{μ_1}. Therefore there is a measure $\tilde{\mu}_1$ (obviously, $S(\tilde{\mu}_1) \cap \operatorname{Int} E_2 = \emptyset$) such that $\mu_1^{\varepsilon_n} \to \tilde{\mu}_1$ and $U^{\mu_1^{\varepsilon_n}} \to U^{\tilde{\mu}_1}$ in $\mathbf{C}$.

Since $U^{\mu_1^{\varepsilon_n}} = U^{\mu_1}$ outside $E_2^{\varepsilon_n}$, we see that $U^{\tilde{\mu}_1} = U^{\mu_1}$ in $\mathbf{C} \backslash E_2$. This implies that $U^{\tilde{\mu}_1} = U^{\mu_1}$ q.e. on ∂E_2. (Indeed, this is true, by definition for any point of ∂E_2 that is regular for $\mathbf{C} \backslash E_2$, and the set of irregular points has zero capacity, by Kellogg's Lemma [La, p. 232]).

It remains to show that $U^{\tilde{\mu}_1} = U^{\mu_1}$ in Int E_2, since then we obtain $U^{\tilde{\mu}_1} = U^{\mu_1}$ q.e. in $\mathbf{C}$, which yields $\tilde{\mu}_1 = \mu_1$.

Let G be any (connected) component of Int E_2. Assuming (i), we see that the set of irregular points of G has μ_1-measure zero. Let $z \in G$, δ_z be the point mass at z, and $\hat{\delta}_z$ be its balayage on ∂G. Since $U^{\delta_z} = U^{\hat{\delta}_z}$ outside G and at regular points of ∂G, we have

$$U^{\mu_1}(z) = \int U^{\delta_z}\, d\mu_1 = \int U^{\hat{\delta}_z}\, d\mu_1 = \int U^{\mu_1}\, d\hat{\delta}_z,$$

the last equality following by Fubini's theorem. Next, as $\hat{\delta}_z$ is C-absolutely continuous (cf. [SaTo, p. 115]) and $U^{\mu_1} = U^{\tilde{\mu}_1}$ q.e. on ∂G, we obtain

$$\int U^{\mu_1}\, d\hat{\delta}_z = \int U^{\tilde{\mu}_1}\, d\hat{\delta}_z = \int U^{\hat{\delta}_z}\, d\tilde{\mu}_1.$$

Finally, since $U^{\hat{\delta}_z} \leqslant U^{\delta_z}$ everywhere, we have

$$\int U^{\hat{\delta}_z}\, d\tilde{\mu}_1 \leqslant \int U^{\delta_z}\, d\tilde{\mu}_1 = U^{\tilde{\mu}_1}(z).$$

It then follows that

$$U^{\mu_1}(z) \leqslant U^{\tilde{\mu}_1}(z), \qquad z \in G.$$

Since we also have $U^{\mu_1} \geqslant U^{\tilde{\mu}_1}$, we conclude that these potentials coincide in G.

Assume now that (ii) holds. Since U^{μ_1} is bounded from above on G and $U^{\tilde{\mu}_1}$ is bounded from below, we see that $U^{\mu_1} - U^{\tilde{\mu}_1}$ is a nonnegative harmonic function in G that is bounded in G and equals 0 q.e. on ∂G. Since G is connected, every point of ∂G is a fine limit point of G. The continuity of potentials in the fine topology then yields, for q.e. $z \in \partial G$,

$$\lim_{\substack{\zeta \to z \\ \zeta \in G}} (U^{\mu_1} - U^{\tilde{\mu}_1})(\zeta) = U^{\mu_1}(z) - U^{\tilde{\mu}_1}(z) = 0.$$

The maximum principle then implies that $U^{\mu_1} = U^{\tilde{\mu}_1}$ in G.

Finally, we show that (iii) $\Rightarrow$ (ii). It suffices to prove that (iii) implies finiteness of U^{μ_1} at every point of ∂E_2, since then the above fine topology argument yields boundedness of U^{μ_1} in any component G of Int E_2. Assume $U^{\mu_1}(z) = \infty$, $z \in \partial E_2$. Then $z \in \partial E_2 \cap S(\mu_1)$. We shall see in the next section that $S(\mu_1) \cap S(\mu_2) = \emptyset$ for any $\mu \in M_\tau^*$. Thus, $U^{\mu_2} \leqslant C$ in some neighborhood of z. Also, $Q \geqslant -C_1$ on ∂E_2. Since

$$U^{\mu_1} - U^{\mu_2} + Q \leqslant -F_2(\mu) \qquad \text{q.e. on } E_2,$$

we conclude that there exists a disk D_z, centered at z, such that cap$(D_z \cap \partial E_2) = 0$ (otherwise, the lower semicontinuity of U^{μ_1} would imply $U^{\mu_1}(z) \leqslant -F_2(\mu) + C + C_1$). But this is impossible, since D_z contains points of $G \subset \text{Int } E_2$ as well as points of $\mathbf{C} \backslash E_2$. Therefore D_z must contain a continuum belonging to ∂E_2, which implies that cap$(D_z \cap \partial E_2) > 0$. ∎

Unfortunately, we were not able to find a satisfactory necessary and sufficient condition for the second assertion of Theorem 3.1 to be true.

4. SOME PROPERTIES OF M_τ^*

Consider the condenser $(S(\mu_1^*), S(\mu_2^*))$, where $\mu^* = \mu_1^* - \mu_2^*$ is the extremal measure for Problem (2.1). Let U^σ be the corresponding condenser potential, that is $\sigma = \sigma_1 - \sigma_2$, $\|\sigma_i\| = 1$, $\sigma_i \geqslant 0$, $S(\sigma_i) \subseteq S(\mu_i^*)$, $i = 1, 2$, while for some constants $a_1 \geqslant 0$, $a_2 \geqslant 0$,

$$U^\sigma = \begin{cases} a_1 & \text{q.e. on } S(\mu_1^*) \\ -a_2 & \text{q.e. on } S(\mu_2^*). \end{cases} \tag{4.1}$$

Therefore

$$-a_2 \leqslant U^\sigma(z) \leqslant a_1, \qquad z \in \overline{\mathbf{C}}. \tag{4.2}$$

Now, by (2.3) of Theorem 2.2, we have

$$\int U^{\mu^*}\,d\sigma=\int (U^{\mu^*}+Q)\,d\sigma-\int Q\,d\sigma=F_1+F_2-\int Q\,d\sigma,$$

while

$$\int U^{\sigma}\,d\mu^*=\tau a_1+(1-\tau)\,a_2$$

by (4.1). The Fubini theorem then yields

$$\tau a_1+(1-\tau)\,a_2=F_1+F_2-\int Q\,d\sigma. \tag{4.3}$$

Next, by the definition of M_τ^*, we have for any $\mu\in M_\tau^*$,

$$\int U^{\mu}\,d\sigma\geqslant F_1(\mu)+F_2(\mu)-\int Q\,d\sigma=F_1+F_2-\int Q\,d\sigma,$$

while

$$\int U^{\sigma}\,d\mu\leqslant\tau a_1+(1-\tau)\,a_2$$

by (4.2). The Fubini theorem and (4.3) then yield

$$\int U^{\sigma}\,d\mu=\tau a_1+(1-\tau)\,a_2, \tag{4.4}$$

$$\int (U^{\mu}+Q-F_1(\mu))\,d\sigma_1-\int (U^{\mu}+Q+F_2(\mu))\,d\sigma_2=0. \tag{4.5}$$

After these preliminaries we turn to the study of M_τ^*.

LEMMA 4.1. *Let $\mu=\mu_1-\mu_2\in M_\tau^*$. Then*

(i) $S(\mu_1)\subseteq S(\sigma_1)\cup\{z\colon U^{\sigma}(z)=a_1\}$, $S(\mu_2)\subseteq S(\sigma_2)\cup\{z\colon U^{\sigma}(z)=-a_2\}$. *In particular*, $S(\mu_1)\cap S(\mu_2)=\varnothing$.

(ii) $U^{\mu}+Q\leqslant F_1(\mu)$ *everywhere on* $S(\sigma_1)$, $U^{\mu}+Q\geqslant -F_2(\mu)$ *everywhere on* $S(\sigma_2)$. *In particular*, U^{μ} *is bounded on* $S(\sigma)$.

(iii)

$$U^{\mu}+Q=\begin{cases}F_1(\mu) & \text{q.e. on } S(\sigma_1)\\ -F_2(\mu) & \text{q.e. on } S(\sigma_2).\end{cases}$$

Proof. Assume $z \notin S(\sigma_1) \cup \{z: U^\sigma(z) = a_1\}$. Then $U^\sigma(z) < a_1$ and $z \notin S(\sigma_1)$. Therefore U^σ is upper semicontinuous in some disk D_z centered at z. This implies that, for some $\varepsilon > 0$, $U^\sigma \leqslant a_1 - \varepsilon$ in D_z. By (4.4), (4.2) we conclude that $\mu_1(D_z) = 0$, so that $z \notin S(\mu_1)$. The proof of the second inclusion in (i) is similar.

Next, since $S(\mu_1) \cap S(\mu_2) = \emptyset$, the function $U^\mu + Q$ is lower (upper) semicontinuous on $S(\sigma_1)$ (on $S(\sigma_2)$). Since the first integrand in (4.5) is non-negative q.e. on $S(\sigma_1)$ and the second is non-positive q.e. on $S(\sigma_2)$, we obtain

$$U^\mu + Q = F_1(\mu), \qquad \sigma_1\text{-a.e. on } S(\sigma_1),$$

$$U^\mu + Q = -F_2(\mu), \qquad \sigma_2\text{-a.e. on } S(\sigma_2).$$

These equalities and the semicontinuity of U^μ and Q, prove (ii). Part (iii) then follows by the definitions of $F_1(\mu)$, $F_2(\mu)$. ∎

COROLLARY 4.2. *Assume that* Int $E_1 =$ Int $E_2 = \emptyset$, *and the complement of* $E_1 \cup E_2$ *is connected. Then* $M_\tau^* = \{\mu^*\}$; *that is, the solution of problem* (2.2) *is unique.*

Proof. Let $\mu = \mu_1 - \mu_2 \in M_\tau^*$. Our assumption implies (via Lemma 4.1(i)):

$$S(\mu_i) \subseteq S(\mu_i^*) = S(\sigma_i), \qquad i = 1, 2. \tag{4.6}$$

Therefore, the potential $U^{\mu - \mu^*}$ is harmonic in $\overline{\mathbf{C}} \setminus S(\mu^*)$ and equals 0 at ∞.

Since μ, $\mu^* \in M_\tau^*$, we have

$$F_1(\mu) - F_1(\mu^*) = -F_2(\mu) + F_2(\mu^*).$$

Then, by Lemma 4.1(iii), we obtain that $U^{\mu - \mu^*} =$ const q.e. on its support.

Next, by Lemma 4.1(ii), U^μ is bounded from above (from below) on $S(\mu_1^*)$ (on $S(\mu_2^*)$). Therefore (see (4.6)), U^μ is bounded on compact subsets of $\mathbf{C}$. So is U^{μ^*}. Hence $U^{\mu - \mu^*}$ is a bounded harmonic function in $\overline{\mathbf{C}} \setminus S(\mu^*)$, which is constant q.e. on $S(\mu^*)$ and 0 at ∞. Therefore it is identically zero, which gives $\mu = \mu^*$. ∎

LEMMA 4.3. *All the measures from* M_τ^* *have the same balayage on* $S(\sigma)$.

Remark. It is not true that they have the same balayage on $S(\mu^*)$. See Example 2.4, in which

$$S(\mu_1^*) = \{z: |z| = 1\}, \qquad S(\mu_2) = \{z: |z| = R_1\} \cup \{z: |z| = R_2\}$$

and there is $\mu \in M_\tau^*$ such that $S(\mu_1) = \{z: |z| = 1\}$, $S(\mu_2) = \{z: |z| = R_2\}$.

Proof of Lemma 4.3. We prove that for any $\mu \in M_\tau^*$, $\hat{\mu} = \hat{\mu}^*$, where the hat symbol stands for the balayage on $S(\sigma)$. It is known, that for some constants c_{1i}, c_{2i}

$$U^{\hat{\mu}_i} = U^{\mu_i} + c_{1i} \qquad \text{q.e. on } S(\sigma_i)$$

$$U^{\hat{\mu}_i^*} = U^{\mu_i^*} + c_{2i} \qquad \text{q.e. on } S(\sigma_i),$$

while $\leqslant$ sign holds everywhere in $\mathbf{C}$. Reasoning as in the proof of Corollary 4.2, we see that $U^{\hat{\mu} - \hat{\mu}^*}$ is a bounded harmonic function in $\overline{\mathbf{C}} \setminus S(\sigma)$, equal to 0 at ∞, and constant q.e. on $S(\sigma)$. Hence $\hat{\mu} = \hat{\mu}^*$. ▮

We have seen (in Example 2.5) that, for $\mu_1 - \mu_2 \in M_\tau^*$, the set $S(\mu_2)$ may intersect E_1, but then "$\inf_{E_1}$"$(U^\mu + Q)$ is attained on $E_1 \setminus S(\mu_2)$. This is a general feature of $\mu \in M_\tau^*$.

LEMMA 4.4. *Let $\mu \in M_\tau^*$, and assume $S(\mu_2) \cap E_1 \neq \varnothing$. Then*

(i) *E_1 is not connected;*

(ii) *E_1 can be decomposed into the union of two disjoint compacta E_1', E_1'', such that*

$$\operatorname{cap} E_1' > 0, \qquad S(\mu_2) \cap E_1' = \varnothing, \qquad E_1'' \neq \varnothing$$

and

$$\text{"}\inf_{E_1'}\text{"}(U^\mu + Q) = F_1(\mu). \tag{4.7}$$

Similar assertions hold for E_2, given that $S(\mu_1) \cap E_2 \neq \varnothing$.

Proof. Fix $\varepsilon > 0$ small enough, so that $E_1^\varepsilon \cap E_2 = \varnothing$ (recall the notation (3.6)). E_1^ε is a finite union of disjoint domains. Let $G_1, ..., G_k$ be those domains for which

$$S(\mu_2) \cap G_i = \varnothing, \qquad i = 1, ..., k,$$

and let $G_{k+1}, ..., G_l$ be the remaining domains, so that

$$S(\mu_2) \cap G_i \neq \varnothing, \qquad i = k+1, ..., l.$$

Let

$$E_1' = E_1 \cap \left(\bigcup_{i=1}^{k} G_i \right), \qquad E_1'' = E_1 \cap \left(\bigcup_{i=k+1}^{l} G_i \right).$$

By assumption, $E_1'' \neq \emptyset$. For $k+1 \leqslant i \leqslant l$, let $K_i := E_1^{\varepsilon/2} \cap G_i$. Then K_i is a compact subset of G_i, and, by assumption, $\mu_2(K_i) > 0$. Let us sweep out that part of μ_2 onto ∂G_i, and let $\tilde{\mu} = \mu_1 - \tilde{\mu}_2$ be the resulting measure.

Inside each G_i we have

$$U^{\tilde{\mu}} = U^{\mu} + \int_{K_i} g_{G_i}(\cdot, t)\, d\mu_2(t), \tag{4.8}$$

where $g_G(\cdot, t)$ stands for the Green function of a domain G with a pole at t. We shall prove in a moment that for any compact $K \subset G$,

$$\inf_{z,\, t \in K} g_G(z, t) \geqslant c > 0, \qquad c = c(K). \tag{4.9}$$

Assuming this, we deduce from (4.8) that

$$\text{“}\inf_{E_1''}\text{”}(U^{\tilde{\mu}} + Q) > F_1(\mu).$$

Also, $U^{\tilde{\mu}} = U^{\mu}$ outside $\bigcup_{i=k+1}^{l} G_i$. In particular this holds on E_2 and on E_1'. Therefore, if we had

$$\operatorname{cap} E_1' = 0 \qquad \text{or} \qquad \text{“}\inf_{E_i'}\text{”}(U^{\mu} + Q) > F_1(\mu),$$

we would obtain that

$$F_1(\tilde{\mu}) + F_2(\tilde{\mu}) > F_1(\mu) + F_2(\mu),$$

contradicting (2.4) of Theorem 2.2 (recall that $\mu \in M_\tau^*$). It thus remains to prove (4.9).

Write $g(z, t) = \log |z-t|^{-1} + u(z, t)$, where $t \in K$ and $u(z, t) = \log |z-t|$ for q.e. z on ∂G. Note that for any $t \in K$, $u(z, t)$ is a *bounded* harmonic function in G (cf. [HaKe, p. 250]).

Since for $t, t' \in K$, $z \in \partial G \backslash I$, cap $I = 0$ we have

$$u(z, t) - u(z, t') = \log \left| \frac{z-t}{z-t'} \right| = O(|t-t'|)$$

uniformly for $t, t' \in K$, $z \in \partial G \backslash I$, the maximum principle yields

$$|u(z, t) - u(z, t')| \leqslant c\, |t-t'|, \qquad z \in G, \quad t, t' \in K.$$

Applying the symmetry of $u(z, t)$ we see that u is jointly continuous on $K \times K$. This immediately gives (4.9). ∎

We also mention a simple property that follows directly from Lemma 4.1(i):

LEMMA 4.5. *Let* $\mu \in M_\tau^*$ *and let* G *be any (connected) component of* $\mathbf{C} \backslash S(\sigma)$. *Then* $\mu_2(G)=0$ *if* $\partial G \subset S(\mu_1)$, *and* $\mu_1(G)=0$ *if* $\partial G \subset S(\mu_2)$. *If* ∂G *contains points of both* $S(\sigma_1)$ *and* $S(\sigma_2)$, *then* $\mu_1(G)=\mu_2(G)=0$.

Finally, we present a result that strengthens part (iv) of Theorem 2.2.

THEOREM 4.6. *For any* $\mu \in M_\tau$ *there holds*

$$\text{“}\inf_{S(\sigma_1)}\text{”}(U^\mu + Q) - \text{“}\sup_{S(\sigma_2)}\text{”}(U^\mu + Q) \leqslant F_1 + F_2. \tag{4.10}$$

If equality holds, and $S(\mu) \subset S(\sigma)$, *then* $\mu = \hat{\mu}^*$ *(the balayage of* μ^* *on* $S(\sigma)$*).*

Proof. We have

$$\tau a_1 + (1-\tau)\, a_2 \geqslant \int U^\sigma \, d\mu = \int U^\mu \, d\sigma$$

$$\geqslant \text{“}\inf_{S(\sigma_1)}\text{”}(U^\mu + Q) - \text{“}\sup_{S(\sigma_2)}\text{”}(U^\mu + Q) - \int Q \, d\sigma.$$

The result now follows, in view of (4.3).

If equality holds and $S(\mu) \subset S(\sigma)$, we consider $U^{\mu - \hat{\mu}^*}$ and, reasoning as in the proof of Corollary 4.2, obtain that $U^{\mu - \hat{\mu}^*} = 0$. ∎

5. CHARACTERIZATION OF WEAK-STAR LIMIT POINTS OF $\nu(R_{mn})$

THEOREM 5.1. *Let* w *be an admissible weight on* $E_1 \cup E_2$, *and let* $\{r_{mn}\}$ *be an asymptotically extremal sequence. Assume* $\nu_{mn} := \nu(r_{mn}) \to \mu$, $(m, n) \in N_\tau$. *Then we have*:

(i) $\mu \in M_\tau^*$, *if* $S(\mu)$ *is compact*;

(ii) $\mu \in \bar{M}_\tau^*$, *if* $S(\mu)$ *is not compact.*

More precisely, let $R > 0$ *be large enough so that the disk* $D_R = \{z\colon |z| < R\}$ *contains* E_1, E_2. *Replace that part of* μ *that sits outside* D_R *by its balayage onto* ∂D_R, *and let* μ^R *denote the resulting measure. Then* $\mu_R \in M_\tau^*$, *and* $\mu^R \to \mu$ *as* $R \to \infty$.

Proof. It is given that

$$\nu_{mn} =: \nu_{mn,1} - \nu_{mn,2} \to \mu_1 - \mu_2 = \mu, \tag{5.1}$$

while

$$F_1(\nu_{mn}) + F_2(\nu_{mn}) \to F_1 + F_2, \qquad (m, n) \in N_\tau. \tag{5.2}$$

(i) If $S(\mu)$ is compact, take R large enough so that the disk D_R contains $S(\mu)$ as well as E_1, E_2. Then ν_{mn} has a mass $o(1)$ on $|z| > R$. Replace this part of μ by its balayage on $|z| = R$. This will add a constant to $U^{\nu_{mn}}$ in D_R, so that (5.2) will hold for a new μ as well. Also, the limit of ν_{mn} remains the same. Thus, we may assume that $S(\nu_{mn}) \subset D_R$, for all $(m, n) \in N_\tau$.

Let $\varepsilon > 0$ be small enough. If we know that

$$\nu_{mn,1}(E_2^\varepsilon) = o(1), \qquad \nu_{mn,2}(E_1^\varepsilon) = o(1), \qquad (m, n) \in N_\tau, \tag{5.3}$$

we proceed as follows. Take the balayage of $\nu_{mn,1}$ (of $\nu_{mn,2}$) from $E_2^\varepsilon (E_1^\varepsilon)$ onto $\partial E_2^\varepsilon (\partial E_1^\varepsilon)$. This will not change the limit measure, while the quantity $F_1(\nu_{mn}) + F_2(\nu_{mn})$ can only increase. Therefore (by (5.2) and Theorem 2.2 (iv)), the relation (5.2) will hold for a new ν_{mn}. But now we have, for $z \in E_1$,

$$U^{\nu_{mn,1}}(z) - U^{\nu_{mn,2}}(z) + Q(z) \geqslant \frac{m}{m+n} \log \frac{1}{R} - \frac{n}{m+n} \log \frac{1}{\varepsilon} + \min_{E_1} Q$$

which implies that the sequence $\{F_1(\nu_{mn})\}$ is bounded from below. Similarly, one can show that $\{F_2(\nu_{mn})\}$ is bounded from below. Then (5.2) implies that these sequences are bounded. Passing to a subsequence, we may assume that for some constant A,

$$F_1(\nu_{mn}) \to F_1 + A, \qquad F_2(\nu_{mn}) \to F_2 - A, \qquad (m, n) \in N_\tau.$$

Since

$$U^{\nu_{mn,2}}(z) \leqslant U^{\nu_{mn,1}}(z) + Q(z) - F_1(\nu_{mn}) \qquad \text{q.e. on } E_1,$$

the lower envelope theorem yields

$$U^{\mu_2}(z) \leqslant U^{\mu_1}(z) + Q(z) - F_1 - A, \qquad \text{q.e. on } E_1.$$

Thus, $F_1(\mu) \geqslant F_1 + A$. Similarly, $F_2(\mu) \geqslant F_2 - A$, and we conclude (by Theorem 2.2(iv)) that $\mu \in M_\tau^*$. If (5.3) does not hold, we apply the same reasoning as in the proof of Lemma 4.4 and show that there exist $E'_1 \subset E_1$, $E'_2 \subset E_2$ such that $F_1(\nu_{mn})$, $F_2(\nu_{mn})$ are attained on E'_1, E'_2 respectively, while

$$\nu_{mn,1}(E'_2) = o(1), \qquad \nu_{mn,2}(E'_1) = o(1), \qquad (m, n) \in N_\tau.$$

Reasoning as above, we conclude that the sequences $\{F_i(\nu_{mn})\}$, $i = 1, 2$, are bounded, and the rest of the proof remains the same.

(ii) If $S(\mu)$ is not compact, take R large enough and replace that part of ν_{mn} that sits on $|z| > R$ by its balayage on $|z| = R$. Let ν^R_{mn} be the resulting measure. Then

$$F_1(\nu^R_{mn}) + F_2(\nu^R_{mn}) = F_1(\nu_{mn}) + F_2(\nu_{mn}) \to F_1 + F_2,$$

and $\nu^R_{mn} \to \mu^R$, $(m, n) \in N_\tau$. By what we have proved, $\mu^R \in M^*_\tau$. The relation $\mu^R \to \mu$ as $R \to \infty$ is obvious. ▮

Corollary 5.2. *Assume* Int E_1 = Int $E_2 = \varnothing$ *and* $\mathbf{C} \backslash (E_1 \cup E_2)$ *is connected. Then for any asymptotically extremal sequence* $\{r_{mn}\}$ *we have* $\nu(r_{mn}) \to \mu^*$.

Proof. Apply Corollary 4.2. ▮

Other corollaries concerning the limit points of $\nu(r_{mn})$ can be drawn, using the results of Section 3. However, we do not have a complete description of all limit points of $\nu(r_{mn})$ in the general case. Yet, we do have the following result, which generalizes the result due to Mhaskar and Saff [MhSa] for the polynomial case.

Theorem 5.3. *Let* $\{r_{mn}\}$ *be asymptotically extremal. Modify* ν_{mn} *in the following way. Let* G *be any connected component of* $\mathbf{C} \backslash S(\sigma)$. *If* $\partial G \subset S(\sigma_1)$ $(\partial G \subset S(\sigma_2))$ *replace* $\nu_{mn,1}$ $(\nu_{mn,2})$ *restricted to* G *by its balayage onto* ∂G. *Then the resulting distribution,* $\tilde{\nu}_{mn}$, *converges weak-star to* $\hat{\mu}^*$, *the balayage of* μ^* *onto* $S(\sigma)$.

Proof. Clearly, we have

$$\text{“}\inf_{S(\sigma_1)}\text{”}(U^{\tilde{\nu}}_{mn} + Q) - \text{“}\sup_{S(\sigma_2)}\text{”}(U^{\tilde{\nu}}_{mn} + Q) \geqslant F_1(\nu_{mn}) + F_2(\nu_{mn}).$$

Note that if G is as above, then $|\tilde{\nu}_{mn}(K)| = o(1)$ for any compact $K \subset G$. Indeed, if $\partial G \subset S(\sigma_i)$, $i = 1$ or $i = 2$, this is true by construction, and otherwise we have $-a_2 < U^\sigma < a_1$ in K, and we may appeal to the beginning of Section 4. Thus, if ν is any weak-star limit of $\{\tilde{\nu}_{mn}\}$, then $S(\nu) \subset S(\sigma)$.

Reasoning as in the proof of Theorem 5.1, we find that for this ν, equality holds in (4.10), and the result follows from Theorem 4.6. ▮

6. THE CASE OF UNBOUNDED SETS

Assume first that only one set, say E_1, is unbounded, and Q satisfies (i), (ii) of Definition 2.1.

Pick $c \notin E_1 \cup E_2$ and apply the Möbius transformation

$$z = c + (\zeta - c)^{-1} =: f(\zeta).$$

This will transform (cf. [LeSa]) problems (2.1), (2.2) into similar ones, with compacta $\tilde{E}_1$, $\tilde{E}_2$ and with Q replaced by

$$\tilde{Q}(\zeta) := Q(f(\zeta)) + (1 - 2\tau) \log \frac{1}{|\zeta - c|}, \qquad c \in \tilde{E}_1.$$

By the assumptions (i), (ii) on Q, $\tilde{Q}$ satisfies these as well, except perhaps at $\zeta = c$. To ensure that $\tilde{Q}$ is lower semicontinuous at c, one can impose the following additional condition on Q:

(iii) $Q(z) + (1 - 2\tau) \log |z|$ is lower semicontinuous at ∞.

If both sets are unbounded (but a positive distance apart) one can impose a stronger condition (cf. [SaTo]) on Q, namely

(iii)′ $Q(z) - \log |z| \to \infty$ as $z \in E_1 \cup E_2 \to \infty$.

This forces μ^* to have a compact support (see [SaTo] for details).

7. APPLICATION TO MINIMAL BLASCHKE PRODUCTS

Let Ω be an arbitrary domain in $\mathbf{C}$ and let $E \subset \Omega$ be compact with cap $E > 0$. Given $w = \exp(-Q)$ (with Q satisfying (i), (ii) of Definition 2.1 on $E_1 := E$), consider the quantity

$$\delta_n(E) := \inf_{B_n} \|B_n w^n\|_E, \tag{7.1}$$

where

$$B_n(z) := \exp\left\{ -\sum_1^n g(z; \alpha_k) - i \sum_1^n \tilde{g}(z; \alpha_k) \right\}, \qquad \alpha_k \in \Omega$$

($\tilde{g}$ stands for the (multiple-valued) conjugate function of the generalized Green function g of Ω).

Let $\nu_n := n^{-1} \sum_{k=1}^{n} \delta_{\alpha_k}$, and let $\hat{\nu}_n$ be the balayage of ν_n onto $\partial\Omega$. It is known that

$$\frac{1}{n} \sum_{1}^{n} g(z; \alpha_k) = U_n^{\nu}(z) - U_n^{\hat{\nu}}(z) + c_n, \tag{7.2}$$

where $c_n = 0$ if Ω is bounded and $c_n = n^{-1} \sum_1^n g(\infty; \alpha_k)$ otherwise.

Thus, if we define Q to be zero on $\partial\Omega$ and set $\mu_n := \nu_n - \tilde{\nu}_n$, we can rewrite (7.1) as

$$\frac{1}{n} \log \frac{1}{\delta_n} = \sup_{\mu_n} \left\{ \inf_E (U^{\mu} + Q) - \sup_{\partial\Omega} (U^{\mu} + Q) \right\}. \tag{7.3}$$

Consider the energy problem

$$V := \inf \left\{ I(\mu) + 2 \int Q \, d\mu \colon \mu = \mu_1 - \mu_2,\ S(\mu_1) \subset E,\ S(\mu_2) \subset \partial\Omega \right\},$$

$$\|\mu_1\| = \|\mu_2\| = 1.$$

This problem coincides with (2.1) for $\tau = 1/2$, except that $\|\mu_i\| = 1$ instead of 1/2.

Let $\mu^* = \mu_1^* - \mu_2^*$ be the equilibrium distribution for this problem. Then [SaTo] μ_2^* coincides with $\hat{\mu}_1^*$, the balayage of μ_1^* onto $\partial\Omega$. With this observation in mind, the following results follow from Theorems 3.1 and 5.3:

(i) $\lim_{n \to \infty} n^{-1} \log(\delta_n^{-1}) = V$;

(ii) Let $\{B_n\}$ be asymptotically extremal. Note that in our case $S(\sigma_2) = \partial\Omega$, $S(\sigma_1) =$ boundary of polynomial convex hull (relative to Ω) of E. Modifying ν_n as usual, we obtain $\hat{\nu}_n \to \hat{\mu}^*$ ($\wedge$ means balayage on $S(\sigma)$).

These assertions should be compared with those of Fisher and Saff [FiSa].

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